The Painlevé Equations and Monodromy Problems

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Basic theme and Background

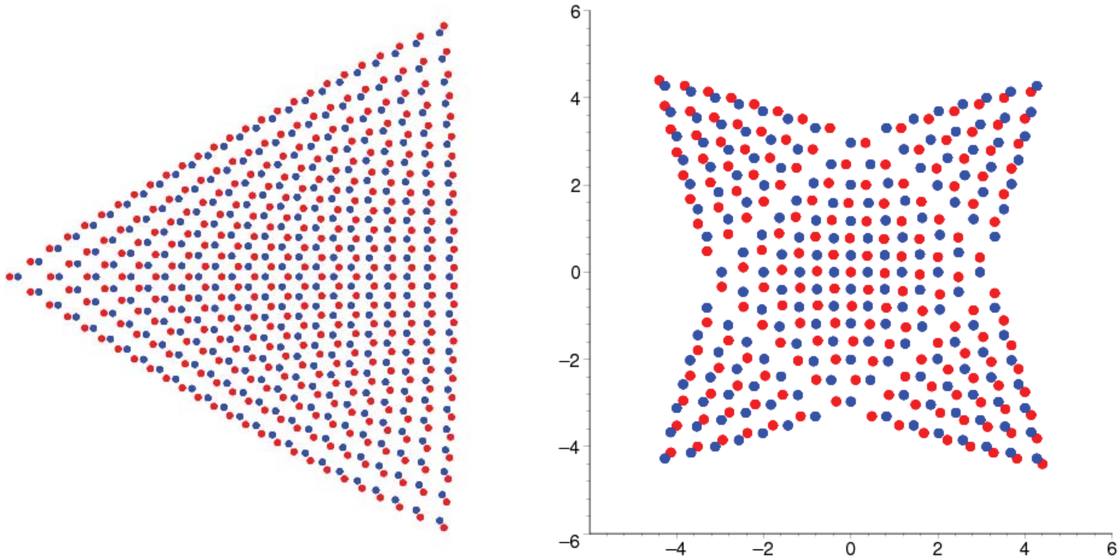
The Painlevé equations, and their solutions, the Painlevé transcendants arise in many disparate parts of pure and applied mathematics and theoretical physics. Painlevé transcendents arise as partition functions in string theories, correlation functions in statistical mechanics, important solutions of differential equations from fluids and general relativity through to Einstein manifolds and monopole moduli spaces in differential geometry, and they arise as generating functions for the topology of moduli spaces of Riemann surfaces and for enumerative problems in algebraic geometry. The Painlevé equations are an integrable system, and therefore have much underlying structure, but, despite their integrability, much of the general theory is still in a somewhat embryonic stage. Indeed the solutions of the Painlevé equations themselves (the so-called Painlevé transcendents) are still some way from being understood as well as the more classical special functions.

Some of the outstanding problems in the field are to fully understand the asymptotics and solve all the corresponding connection problems of the Painlevé transcendents: for example if one has a Painlevé transcendent with a certain behaviour at zero, then can we say how it behaves at infinity? More generally, the Painlevé equations can be viewed as the simplest cases of equations controlling monodromy preserving deformations of linear differential operators on the Riemann sphere and one can ask the same questions for any such (nonlinear) isomonodromy equations.

Some less concrete problems arise in understanding the various applications the Painlevé equations have found, for example in Random matrix ensembles or the Tracey-Widom distribution controlling the largest increasing subsequences of random permutations. One can try for example to see directly an isomonodromic deformation in the original problem (which would explain clearly why we expected to find a Painlevé solution in the answer).

Structure

The month-long programme brought together three communities of researchers: the Japanese school, who are strongly influenced by methods from algebraic geometry and infinite group theory, members of the Enigma Marie Curie Research Training Network, the European Network in Geometry, Mathematical Physics and its Applications, and leading UK researchers. The Japanese school were partially supported by the Japan Society for the Promotion of Science, and the Enigma Network supported the participation of many of its members. The programme was



Left: The poles of a rational solution of the second Painlevé equation. Those with residue +1 are in blue and those with residue -1 in red. Right: The zeroes (red) and poles (blue) of a rational-oscillatory solution of the de-focussing nonlinear Schrodinger equation.

focused around the two workshops in the middle two weeks of the month. The first workshop was designed for younger people and aimed to cover the basic theory and background results in the area, whilst the second was a research workshop aimed more at disseminating recent results by active workers in the field.

The introductory workshop consisted of two longer courses each of four hours together with seven shorter courses of two hours each, plus a scattering of eight half-hour talks given by younger participants and a poster session consisting of a number of posters. The long courses (given by Professors Okamoto and Umemura) gave the main introduction to the Painlevé equation and to their differential Galois theory. The two-hour courses covered other topics centred on the Painlevé equations such as: their asymptotics, their special solutions, their relation to random matrices, their discrete analogues, their Hamiltonian structure and their relation to Riemann-Hilbert problems.

The research workshop brought together an outstanding collection of researchers. The program consisted of twenty-two one hour talks plus eight half-hour presentations and a poster session. The main topics covered in the lectures fit into the following categories: Links to integrable hierarchies, Random matrices, Behaviour of Painlevé transcendents (and isomonodromic deformations), Links to 2D integrable quantum field theory and Frobenius manifolds, Discrete Painlevé equations (discrete analytic functions, rational algebraic surfaces, orthogonal polynomials), and nonlinear differential Galois theory.

Outcomes

The introductory courses gave both an excellent introduction to modern research in the field for graduate students, but also allowed experts on one field to learn about progress in neighbouring fields. In their reports, the participants all mention important input into their research arising both from the lectures and from their interactions with other participants. Many of these interactions have led to further collaborations and papers. We just mention a few of these collaborations and papers here. Nalini Joshi advanced her collaboration with Marta Mazzocco and Frank Nijhoff, finishing a joint paper with the latter on *A Lax Pair for a lattice modified KdV equation* and initiated a new collaboration with Clarkson. Clarkson worked on papers with Shirley Harris on *Painlevé analysis and similarity reductions for the Magma equation* and with Galina Filipuk on *The fourth Painlevé hierarchy and associated special polynomials*. Casale, Malgrange and Umemura managed to prove the foundational result of the equivalence of the two nonlinear differential Galois theories of Umemura and Malgranges with surprising applications to the Picard solution of the sixth Painlevé equation. Veselov, Felder and Feigin essentially completed the first draft of a paper on the geometry of V-systems describing logarithmic Frobenius structures. Veselov also intrigued many participants with the observation that an old mathematical recreation of Roger Penrose on iterated mappings was effectively a specialization of the discrete Painlevé equations. Overall, participants seem to have learnt a lot, had many useful interactions and universally expressed the wish that the programme had been longer.

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